

The λ -calculus: from simple types to non-idempotent intersection types

Day 5: More about non-idempotent intersection types for the λ -calculus

Giulio Guerrieri

Department of Informatics, University of Sussex (Brighton, UK)

✉ g.guerrieri@sussex.ac.uk 🌐 <https://pageperso.lis-lab.fr/~giulio.guerrieri/>

37th Escuela de Ciencias Informáticas (ECI 2024)

Buenos Aires (Argentina), 2 August 2024

Outline

- 1 Characterizing leftmost-outermost normalization in NI
- 2 Some final remarks about non-idempotent intersection types
- 3 Conclusion, exercises and bibliography

Outline

- 1 Characterizing leftmost-outermost normalization in NI
- 2 Some final remarks about non-idempotent intersection types
- 3 Conclusion, exercises and bibliography

What can we do with non-idempotent intersection types?

Goal. We want to characterize all and only the $\ell\beta$ -normalizing terms via NI.

Motivation 1. $\rightarrow_{\ell\beta}$ is a normalizing strategy for $\rightarrow_{\beta}$: reaches a β -normal form if it exists.

Motivation 2. The number of $\rightarrow_{\ell\beta}$ steps is a reasonable cost model.

Bonus. We use the **same** type system NI (same rules), we just consider specific types.

~ NI is versatile. Also, some results are already proven and can be used immediately.

To achieve this **qualitative** characterization, we need to prove two properties.

- 1 **Correctness:** if a term is typable in NI with specific types then it is $\ell\beta$ -normalizing.
- 2 **Completeness:** if a term is $\ell\beta$ -normalizing then it is typable in NI with specific types.

Bonus. We can extract some **quantitative** information from NI about:

- 1 the **length of evaluation** (the number of $\ell\beta$ -steps to reach the β -normal form);
- 2 the **size of the output** (i.e. of the $\ell\beta$ -normal form).

What can we do with non-idempotent intersection types?

Goal. We want to characterize all and only the $\ell\beta$ -normalizing terms via NI.

Motivation 1. $\rightarrow_{\ell\beta}$ is a normalizing strategy for $\rightarrow_{\beta}$: reaches a β -normal form if it exists.

Motivation 2. The number of $\rightarrow_{\ell\beta}$ steps is a reasonable cost model.

Bonus. We use the **same** type system NI (same rules), we just consider specific types.

↪ NI is versatile. Also, some results are already proven and can be used immediately.

To achieve this **qualitative** characterization, we need to prove two properties.

- 1 **Correctness:** if a term is typable in NI with specific types then it is $\ell\beta$ -normalizing.
- 2 **Completeness:** if a term is $\ell\beta$ -normalizing then it is typable in NI with specific types.

Bonus. We can extract some **quantitative** information from NI about:

- 1 the **length of evaluation** (the number of $\ell\beta$ -steps to reach the β -normal form);
- 2 the **size of the output** (i.e. of the $\ell\beta$ -normal form).

What can we do with non-idempotent intersection types?

Goal. We want to characterize all and only the $\ell\beta$ -normalizing terms via NI.

Motivation 1. $\rightarrow_{\ell\beta}$ is a normalizing strategy for $\rightarrow_{\beta}$: reaches a β -normal form if it exists.

Motivation 2. The number of $\rightarrow_{\ell\beta}$ steps is a reasonable cost model.

Bonus. We use the **same** type system NI (same rules), we just consider specific types.

$\rightsquigarrow$ NI is versatile. Also, some results are already proven and can be used immediately.

To achieve this **qualitative** characterization, we need to prove two properties.

- 1 **Correctness:** if a term is typable in NI with specific types then it is $\ell\beta$ -normalizing.
- 2 **Completeness:** if a term is $\ell\beta$ -normalizing then it is typable in NI with specific types.

Bonus. We can extract some **quantitative** information from NI about:

- 1 the **length of evaluation** (the number of $\ell\beta$ -steps to reach the β -normal form);
- 2 the **size of the output** (i.e. of the $\ell\beta$ -normal form).

What can we do with non-idempotent intersection types?

Goal. We want to characterize all and only the $\ell\beta$ -normalizing terms via NI.

Motivation 1. $\rightarrow_{\ell\beta}$ is a normalizing strategy for $\rightarrow_{\beta}$: reaches a β -normal form if it exists.

Motivation 2. The number of $\rightarrow_{\ell\beta}$ steps is a reasonable cost model.

Bonus. We use the **same** type system NI (same rules), we just consider specific types.

$\rightsquigarrow$ NI is versatile. Also, some results are already proven and can be used immediately.

To achieve this **qualitative** characterization, we need to prove two properties.

- ❶ **Correctness:** if a term is typable in NI with specific types then it is $\ell\beta$ -normalizing.
- ❷ **Completeness:** if a term is $\ell\beta$ -normalizing then it is typable in NI with specific types.

Bonus. We can extract some **quantitative** information from NI about:

- ❶ the **length of evaluation** (the number of $\ell\beta$ -steps to reach the β -normal form);
- ❷ the **size of the output** (i.e. of the $\ell\beta$ -normal form).

A way to identify some specific types: positive and negative occurrences

Def. The sets $\text{oc}_+(T)$ and $\text{oc}_-(T)$ **positive** and **negative** occurrences of a type T are:

$$\begin{array}{c}
 \frac{}{A \in \text{oc}_+(A)} \qquad \frac{T \in \text{oc}_-(M) \text{ or } T \in \text{oc}_+(A)}{T \in \text{oc}_+(M \multimap A)} \qquad \frac{T \in \text{oc}_+(M) \text{ or } T \in \text{oc}_-(A)}{T \in \text{oc}_-(M \multimap A)} \\
 \\
 \frac{}{M \in \text{oc}_+(M)} \qquad \frac{\exists A \in M : T \in \text{oc}_+(A)}{T \in \text{oc}_+(M)} \qquad \frac{\exists A \in M : T \in \text{oc}_-(A)}{T \in \text{oc}_-(M)} \\
 \\
 \frac{T \in \text{oc}_+(\Gamma) \text{ or } T \in \text{oc}_+(M)}{T \in \text{oc}_+(\Gamma, x : M)} \qquad \frac{T \in \text{oc}_-(\Gamma) \text{ or } T \in \text{oc}_-(M)}{T \in \text{oc}_-(\Gamma, x : M)}
 \end{array}$$

Ex. $[] \in \text{oc}_-([], \multimap A)$, $[] \in \text{oc}_-([[], \multimap A, A])$, $[] \in \text{oc}_-(x : [[], \multimap A])$, $[] \in \text{oc}_+([[], \multimap A] \multimap A)$.

$[] \in \text{oc}_7([], \multimap A) :$



$[] \in \text{oc}_-([], \multimap A)$

$[] \in \text{oc}_+([[], \multimap A] \multimap A) :$



$[] \in \text{oc}_+([[], \multimap A] \multimap A)$

Ex. Let $A = [[], \multimap [[], \multimap X]] \multimap [[], \multimap X]$, then $[] \in \text{oc}_+(A) \cap \text{oc}_-(A)$.

A way to identify some specific types: positive and negative occurrences

Def. The sets $\text{oc}_+(T)$ and $\text{oc}_-(T)$ **positive** and **negative** occurrences of a type T are:

$$\begin{array}{c}
 \frac{}{A \in \text{oc}_+(A)} \qquad \frac{T \in \text{oc}_-(M) \text{ or } T \in \text{oc}_+(A)}{T \in \text{oc}_+(M \multimap A)} \qquad \frac{T \in \text{oc}_+(M) \text{ or } T \in \text{oc}_-(A)}{T \in \text{oc}_-(M \multimap A)} \\
 \\
 \frac{}{M \in \text{oc}_+(M)} \qquad \frac{\exists A \in M : T \in \text{oc}_+(A)}{T \in \text{oc}_+(M)} \qquad \frac{\exists A \in M : T \in \text{oc}_-(A)}{T \in \text{oc}_-(M)} \\
 \\
 \frac{T \in \text{oc}_+(\Gamma) \text{ or } T \in \text{oc}_+(M)}{T \in \text{oc}_+(\Gamma, x : M)} \qquad \frac{T \in \text{oc}_-(\Gamma) \text{ or } T \in \text{oc}_-(M)}{T \in \text{oc}_-(\Gamma, x : M)}
 \end{array}$$

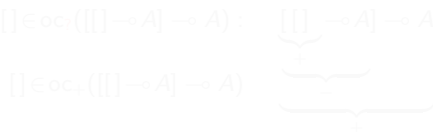
Ex. $[] \in \text{oc}_-([] \multimap A)$, $[] \in \text{oc}_-([[] \multimap A, A])$, $[] \in \text{oc}_-(x : [[] \multimap A])$, $[] \in \text{oc}_+([[] \multimap A] \multimap A)$.

$[] \in \text{oc}_7([] \multimap A) :$



$[] \in \text{oc}_-([] \multimap A)$

$[] \in \text{oc}_+([[] \multimap A] \multimap A) :$



$[] \in \text{oc}_+([[] \multimap A] \multimap A)$

Ex. Let $A = [[] \multimap [] \multimap X] \multimap [] \multimap X$, then $[] \in \text{oc}_+(A) \cap \text{oc}_-(A)$.

A way to identify some specific types: positive and negative occurrences

Def. The sets $\text{oc}_+(T)$ and $\text{oc}_-(T)$ **positive** and **negative** occurrences of a type T are:

$$\begin{array}{c}
 \frac{}{A \in \text{oc}_+(A)} \qquad \frac{T \in \text{oc}_-(M) \text{ or } T \in \text{oc}_+(A)}{T \in \text{oc}_+(M \multimap A)} \qquad \frac{T \in \text{oc}_+(M) \text{ or } T \in \text{oc}_-(A)}{T \in \text{oc}_-(M \multimap A)} \\
 \\
 \frac{}{M \in \text{oc}_+(M)} \qquad \frac{\exists A \in M : T \in \text{oc}_+(A)}{T \in \text{oc}_+(M)} \qquad \frac{\exists A \in M : T \in \text{oc}_-(A)}{T \in \text{oc}_-(M)} \\
 \\
 \frac{T \in \text{oc}_+(\Gamma) \text{ or } T \in \text{oc}_+(M)}{T \in \text{oc}_+(\Gamma, x : M)} \qquad \frac{T \in \text{oc}_-(\Gamma) \text{ or } T \in \text{oc}_-(M)}{T \in \text{oc}_-(\Gamma, x : M)}
 \end{array}$$

Ex. $[] \in \text{oc}_-([] \multimap A)$, $[] \in \text{oc}_-([[] \multimap A, A])$, $[] \in \text{oc}_-(x : [[] \multimap A])$, $[] \in \text{oc}_+([[] \multimap A] \multimap A)$.

$$\begin{array}{cc}
 [] \in \text{oc}_?([] \multimap A) : \underbrace{[]}_{+} \multimap A & [] \in \text{oc}_?([[] \multimap A] \multimap A) : \underbrace{[[] \multimap A]}_{+} \multimap A \\
 [] \in \text{oc}_-([] \multimap A) \quad \underbrace{\hspace{1.5cm}}_{-} & [] \in \text{oc}_+([[] \multimap A] \multimap A) \quad \underbrace{\hspace{1.5cm}}_{-} \\
 & \underbrace{\hspace{1.5cm}}_{+}
 \end{array}$$

Ex. Let $A = [[] \multimap [[] \multimap X] \multimap [[] \multimap X]$, then $[] \in \text{oc}_+(A) \cap \text{oc}_-(A)$.

A way to identify some specific types: positive and negative occurrences

Def. The sets $\text{oc}_+(T)$ and $\text{oc}_-(T)$ **positive** and **negative** occurrences of a type T are:

$$\begin{array}{c}
 \frac{}{A \in \text{oc}_+(A)} \qquad \frac{T \in \text{oc}_-(M) \text{ or } T \in \text{oc}_+(A)}{T \in \text{oc}_+(M \multimap A)} \qquad \frac{T \in \text{oc}_+(M) \text{ or } T \in \text{oc}_-(A)}{T \in \text{oc}_-(M \multimap A)} \\
 \\
 \frac{}{M \in \text{oc}_+(M)} \qquad \frac{\exists A \in M : T \in \text{oc}_+(A)}{T \in \text{oc}_+(M)} \qquad \frac{\exists A \in M : T \in \text{oc}_-(A)}{T \in \text{oc}_-(M)} \\
 \\
 \frac{T \in \text{oc}_+(\Gamma) \text{ or } T \in \text{oc}_+(M)}{T \in \text{oc}_+(\Gamma, x : M)} \qquad \frac{T \in \text{oc}_-(\Gamma) \text{ or } T \in \text{oc}_-(M)}{T \in \text{oc}_-(\Gamma, x : M)}
 \end{array}$$

Ex. $[] \in \text{oc}_-([] \multimap A)$, $[] \in \text{oc}_-([[] \multimap A, A])$, $[] \in \text{oc}_-(x : [[] \multimap A])$, $[] \in \text{oc}_+([[] \multimap A] \multimap A)$.

$$\begin{array}{cc}
 [] \in \text{oc}_?([] \multimap A) : \underbrace{[]}_{+} \multimap A & [] \in \text{oc}_?([[] \multimap A] \multimap A) : \underbrace{[[] \multimap A]}_{+} \multimap A \\
 [] \in \text{oc}_-([] \multimap A) \quad \underbrace{\hspace{1.5cm}}_{-} & [] \in \text{oc}_+([[] \multimap A] \multimap A) \quad \underbrace{\underbrace{\hspace{1.5cm}}_{-}}_{+}
 \end{array}$$

Ex. Let $A = [[] \multimap [] \multimap X] \multimap [] \multimap X$, then $[] \in \text{oc}_+(A) \cap \text{oc}_-(A)$.

A way to identify some specific types: positive and negative occurrences

Def. The sets $\text{oc}_+(T)$ and $\text{oc}_-(T)$ **positive** and **negative** occurrences of a type T are:

$$\begin{array}{c}
 \frac{}{A \in \text{oc}_+(A)} \qquad \frac{T \in \text{oc}_-(M) \text{ or } T \in \text{oc}_+(A)}{T \in \text{oc}_+(M \multimap A)} \qquad \frac{T \in \text{oc}_+(M) \text{ or } T \in \text{oc}_-(A)}{T \in \text{oc}_-(M \multimap A)} \\
 \\
 \frac{}{M \in \text{oc}_+(M)} \qquad \frac{\exists A \in M : T \in \text{oc}_+(A)}{T \in \text{oc}_+(M)} \qquad \frac{\exists A \in M : T \in \text{oc}_-(A)}{T \in \text{oc}_-(M)} \\
 \\
 \frac{T \in \text{oc}_+(\Gamma) \text{ or } T \in \text{oc}_+(M)}{T \in \text{oc}_+(\Gamma, x : M)} \qquad \frac{T \in \text{oc}_-(\Gamma) \text{ or } T \in \text{oc}_-(M)}{T \in \text{oc}_-(\Gamma, x : M)}
 \end{array}$$

Ex. $[] \in \text{oc}_-([], \multimap A)$, $[] \in \text{oc}_-([[], \multimap A, A])$, $[] \in \text{oc}_-(x : [[], \multimap A])$, $[] \in \text{oc}_+([[], \multimap A] \multimap A)$.

$$\begin{array}{ccc}
 [] \in \text{oc}_?([], \multimap A) : & \underbrace{[]}_{+} \multimap A & [] \in \text{oc}_?([[], \multimap A] \multimap A) : \underbrace{[[]]_{+} \multimap A}_{-} \multimap A \\
 [] \in \text{oc}_-([], \multimap A) & \underbrace{\quad}_{-} & [] \in \text{oc}_+([[], \multimap A] \multimap A) & \underbrace{\quad}_{+}
 \end{array}$$

Ex. Let $A = [[], \multimap [[], \multimap X]] \multimap [[], \multimap X]$, then $[] \in \text{oc}_+(A) \cap \text{oc}_-(A)$.

A way to identify some specific types: positive and negative occurrences

Def. The sets $\text{oc}_+(T)$ and $\text{oc}_-(T)$ **positive** and **negative** occurrences of a type T are:

$$\begin{array}{c}
 \frac{}{A \in \text{oc}_+(A)} \qquad \frac{T \in \text{oc}_-(M) \text{ or } T \in \text{oc}_+(A)}{T \in \text{oc}_+(M \multimap A)} \qquad \frac{T \in \text{oc}_+(M) \text{ or } T \in \text{oc}_-(A)}{T \in \text{oc}_-(M \multimap A)} \\
 \\
 \frac{}{M \in \text{oc}_+(M)} \qquad \frac{\exists A \in M : T \in \text{oc}_+(A)}{T \in \text{oc}_+(M)} \qquad \frac{\exists A \in M : T \in \text{oc}_-(A)}{T \in \text{oc}_-(M)} \\
 \\
 \frac{T \in \text{oc}_+(\Gamma) \text{ or } T \in \text{oc}_+(M)}{T \in \text{oc}_+(\Gamma, x : M)} \qquad \frac{T \in \text{oc}_-(\Gamma) \text{ or } T \in \text{oc}_-(M)}{T \in \text{oc}_-(\Gamma, x : M)}
 \end{array}$$

Ex. $[] \in \text{oc}_-([], \multimap A)$, $[] \in \text{oc}_-([[], \multimap A, A])$, $[] \in \text{oc}_-(x : [[], \multimap A])$, $[] \in \text{oc}_+([[], \multimap A] \multimap A)$.

$$\begin{array}{cc}
 [] \in \text{oc}_?([], \multimap A) : \underbrace{[]}_{+} \multimap A & [] \in \text{oc}_?([[], \multimap A] \multimap A) : \underbrace{[[]]_{+} \multimap A}_{-} \multimap A \\
 [] \in \text{oc}_-([], \multimap A) & [] \in \text{oc}_+([[], \multimap A] \multimap A) : \underbrace{\underbrace{[[]]_{+} \multimap A}_{-}}_{+} \multimap A
 \end{array}$$

Ex. Let $A = [[], \multimap [[], \multimap X]] \multimap [[], \multimap X]$, then $[] \in \text{oc}_+(A) \cap \text{oc}_-(A)$.

A way to identify some specific types: positive and negative occurrences

Def. The sets $\text{oc}_+(T)$ and $\text{oc}_-(T)$ **positive** and **negative** occurrences of a type T are:

$$\begin{array}{c}
 \frac{}{A \in \text{oc}_+(A)} \qquad \frac{T \in \text{oc}_-(M) \text{ or } T \in \text{oc}_+(A)}{T \in \text{oc}_+(M \multimap A)} \qquad \frac{T \in \text{oc}_+(M) \text{ or } T \in \text{oc}_-(A)}{T \in \text{oc}_-(M \multimap A)} \\
 \\
 \frac{}{M \in \text{oc}_+(M)} \qquad \frac{\exists A \in M : T \in \text{oc}_+(A)}{T \in \text{oc}_+(M)} \qquad \frac{\exists A \in M : T \in \text{oc}_-(A)}{T \in \text{oc}_-(M)} \\
 \\
 \frac{T \in \text{oc}_+(\Gamma) \text{ or } T \in \text{oc}_+(M)}{T \in \text{oc}_+(\Gamma, x : M)} \qquad \frac{T \in \text{oc}_-(\Gamma) \text{ or } T \in \text{oc}_-(M)}{T \in \text{oc}_-(\Gamma, x : M)}
 \end{array}$$

Ex. $[] \in \text{oc}_-([], \multimap A)$, $[] \in \text{oc}_-([[], \multimap A, A])$, $[] \in \text{oc}_-(x : [[], \multimap A])$, $[] \in \text{oc}_+([[], \multimap A] \multimap A)$.

$$\begin{array}{cc}
 [] \in \text{oc}_?([], \multimap A) : \underbrace{[] \multimap A}_{+} & [] \in \text{oc}_?([[], \multimap A] \multimap A) : \underbrace{\underbrace{[[], \multimap A] \multimap A}_{+}}_{-} \\
 [] \in \text{oc}_-([], \multimap A) & [] \in \text{oc}_+([[], \multimap A] \multimap A) : \underbrace{\underbrace{\quad}_{-}}_{+}
 \end{array}$$

Ex. Let $A = [[], \multimap [[], \multimap X] \multimap [[], \multimap X]$, then $[] \in \text{oc}_+(A) \cap \text{oc}_-(A)$.

Some specific types in NI: shrinking

- ❶ A linear type A is **shrinking** if $|M| \geq 1$ for all $M \in \text{oc}_+(A)$.
- ❷ A linear type A is **co-shrinking** if $|M| \geq 1$ for all $M \in \text{oc}_-(A)$.
- ❸ A multi type M is **shrinking** (resp. **co-shrinking**) if so is every $A \in M$.
- ❹ An environment Γ is **co-shrinking** if so is $\Gamma(x)$ for every variable x .
- ❺ A derivation $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : T$ is **shrinking** if Γ is co-shrinking and T is shrinking.

Ex. $[[] \multimap X, X]$ is only shrinking, $[[] \multimap X] \multimap X$ is only co-shrinking, X is both.

Rmk. $M \multimap A$ is co-shrinking if and only if $M \neq []$ is shrinking and A is co-shrinking.

Rmk. $M \multimap A$ is shrinking if and only if M is co-shrinking and A is shrinking.

Rmk. Let $\Gamma = \Gamma_1 \uplus \Gamma_2$: Γ is co-shrinking if and only if so are Γ_1 and Γ_2 .

Idea: To guarantee that B is shrinking in $\Gamma \vdash s : B$, we need Γ to be co-shrinking.

$$\frac{\Gamma, x : M \vdash t : A}{\Gamma \vdash \lambda x. t : \underbrace{M \multimap A}_{\text{shrinking}}} \lambda$$

Some specific types in NI: shrinking

- ❶ A linear type A is **shrinking** if $|M| \geq 1$ for all $M \in \text{oc}_+(A)$.
- ❷ A linear type A is **co-shrinking** if $|M| \geq 1$ for all $M \in \text{oc}_-(A)$.
- ❸ A multi type M is **shrinking** (resp. **co-shrinking**) if so is every $A \in M$.
- ❹ An environment Γ is **co-shrinking** if so is $\Gamma(x)$ for every variable x .
- ❺ A derivation $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : T$ is **shrinking** if Γ is co-shrinking and T is shrinking.

Ex. $[[] \multimap X, X]$ is only shrinking, $[[] \multimap X] \multimap X$ is only co-shrinking, X is both.

Rmk. $M \multimap A$ is co-shrinking if and only if $M \neq []$ is shrinking and A is co-shrinking.

Rmk. $M \multimap A$ is shrinking if and only if M is co-shrinking and A is shrinking.

Rmk. Let $\Gamma = \Gamma_1 \uplus \Gamma_2$: Γ is co-shrinking if and only if so are Γ_1 and Γ_2 .

Idea: To guarantee that B is shrinking in $\Gamma \vdash s : B$, we need Γ to be co-shrinking.

$$\frac{\Gamma, x : M \vdash t : A}{\Gamma \vdash \lambda x. t : \underbrace{M \multimap A}_{\text{shrinking}}} \lambda$$

Some specific types in NI: shrinking

- ❶ A linear type A is **shrinking** if $|M| \geq 1$ for all $M \in \text{oc}_+(A)$.
- ❷ A linear type A is **co-shrinking** if $|M| \geq 1$ for all $M \in \text{oc}_-(A)$.
- ❸ A multi type M is **shrinking** (resp. **co-shrinking**) if so is every $A \in M$.
- ❹ An environment Γ is **co-shrinking** if so is $\Gamma(x)$ for every variable x .
- ❺ A derivation $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : T$ is **shrinking** if Γ is co-shrinking and T is shrinking.

Ex. $[[] \multimap X, X]$ is only shrinking, $[[] \multimap X] \multimap X$ is only co-shrinking, X is both.

Rmk. $M \multimap A$ is co-shrinking if and only if $M \neq []$ is shrinking and A is co-shrinking.

Rmk. $M \multimap A$ is shrinking if and only if M is co-shrinking and A is shrinking.

Rmk. Let $\Gamma = \Gamma_1 \uplus \Gamma_2$: Γ is co-shrinking if and only if so are Γ_1 and Γ_2 .

Idea: To guarantee that B is shrinking in $\Gamma \vdash s : B$, we need Γ to be co-shrinking.

$$\frac{\Gamma, x : M \vdash t : A}{\Gamma \vdash \lambda x. t : \underbrace{M \multimap A}_{\text{shrinking}}} \lambda$$

Some specific types in NI: shrinking

- ❶ A linear type A is **shrinking** if $|M| \geq 1$ for all $M \in \text{oc}_+(A)$.
- ❷ A linear type A is **co-shrinking** if $|M| \geq 1$ for all $M \in \text{oc}_-(A)$.
- ❸ A multi type M is **shrinking** (resp. **co-shrinking**) if so is every $A \in M$.
- ❹ An environment Γ is **co-shrinking** if so is $\Gamma(x)$ for every variable x .
- ❺ A derivation $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : T$ is **shrinking** if Γ is co-shrinking and T is shrinking.

Ex. $[[] \multimap X, X]$ is only shrinking, $[[] \multimap X] \multimap X$ is only co-shrinking, X is both.

Rmk. $M \multimap A$ is co-shrinking if and only if $M \neq []$ is shrinking and A is co-shrinking.

Rmk. $M \multimap A$ is shrinking if and only if M is co-shrinking and A is shrinking.

Rmk. Let $\Gamma = \Gamma_1 \uplus \Gamma_2$: Γ is co-shrinking if and only if so are Γ_1 and Γ_2 .

Idea: To guarantee that B is shrinking in $\Gamma \vdash s : B$, we need Γ to be co-shrinking.

$$\frac{\Gamma, x : M \vdash t : A}{\Gamma \vdash \lambda x. t : \underbrace{M \multimap A}_{\text{shrinking}}} \lambda$$

Some specific types in NI: shrinking

- ❶ A linear type A is **shrinking** if $|M| \geq 1$ for all $M \in \text{oc}_+(A)$.
- ❷ A linear type A is **co-shrinking** if $|M| \geq 1$ for all $M \in \text{oc}_-(A)$.
- ❸ A multi type M is **shrinking** (resp. **co-shrinking**) if so is every $A \in M$.
- ❹ An environment Γ is **co-shrinking** if so is $\Gamma(x)$ for every variable x .
- ❺ A derivation $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : T$ is **shrinking** if Γ is co-shrinking and T is shrinking.

Ex. $[[] \multimap X, X]$ is only shrinking, $[[] \multimap X] \multimap X$ is only co-shrinking, X is both.

Rmk. $M \multimap A$ is co-shrinking if and only if $M \neq []$ is shrinking and A is co-shrinking.

Rmk. $M \multimap A$ is shrinking if and only if M is co-shrinking and A is shrinking.

Rmk. Let $\Gamma = \Gamma_1 \uplus \Gamma_2$: Γ is co-shrinking if and only if so are Γ_1 and Γ_2 .

Idea: To guarantee that B is shrinking in $\Gamma \vdash s : B$, we need Γ to be co-shrinking.

$$\frac{\Gamma, x : \overbrace{M}^{\text{co-shrinking}} \vdash t : \overbrace{A}^{\text{shrinking}}}{\Gamma \vdash \lambda x. t : \underbrace{M \multimap A}_{\text{shrinking}}} \lambda$$

Ingredients to prove correctness

Def. The **size** $|t|$ of a term t is defined by induction on t as follows:

$$|x| = 1$$

$$|\lambda x.t| = 1 + |t|$$

$$|st| = 1 + |s| + |t|$$

Lemma (Spreading of shrinkingness)

Let $t \neq \lambda x.s$ be β -normal and $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$. If Γ is co-shrinking then A is co-shrinking.

Proof. By induction on $m \in \mathbb{N}$, as $t = xt_1 \dots t_m$ for some $m \in \mathbb{N}$, β -normal $t_1, \dots, t_m$. $\square$

Lemma (Typing β -normal forms in a co-shrinking environment)

Let t be β -normal and $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$. If Γ is co-shrinking and (A is shrinking or t is not an abstraction), then $|t| \leq |\mathcal{D}|$.

Proof. Every β -normal term is of the form $t = \lambda x_n \dots \lambda x_1.yt_1 \dots t_m$ for some $m, n \in \mathbb{N}$, with $t_1, \dots, t_m$ β -normal. The lemma is proved by induction on $|t| \in \mathbb{N}$. We use the lemma above if $n = 0$ and $m > 0$. $\square$

Ingredients to prove correctness

Def. The **size** $|t|$ of a term t is defined by induction on t as follows:

$$|x| = 1$$

$$|\lambda x.t| = 1 + |t|$$

$$|st| = 1 + |s| + |t|$$

Lemma (Spreading of shrinkingness)

Let $t \neq \lambda x.s$ be β -normal and $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$. If Γ is co-shrinking then A is co-shrinking.

Proof. By induction on $m \in \mathbb{N}$, as $t = xt_1 \dots t_m$ for some $m \in \mathbb{N}$, β -normal $t_1, \dots, t_m$. $\square$

Lemma (Typing β -normal forms in a co-shrinking environment)

Let t be β -normal and $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$. If Γ is co-shrinking and (A is shrinking or t is not an abstraction), then $|t| \leq |\mathcal{D}|$.

Proof. Every β -normal term is of the form $t = \lambda x_n \dots \lambda x_1.yt_1 \dots t_m$ for some $m, n \in \mathbb{N}$, with $t_1, \dots, t_m$ β -normal. The lemma is proved by induction on $|t| \in \mathbb{N}$. We use the lemma above if $n = 0$ and $m > 0$. $\square$

Correctness of NI: shrinking typability implies $\ell\beta$ -normalization

Proposition (Quantitative subject reduction for shrinking derivations)

If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ is shrinking and $t \rightarrow_{\ell\beta} t'$, then there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \rightarrow_{\ell\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \rightarrow_{\ell\beta} u\{s/x\} = t'$: so, $\mathcal{D}$ must have the form below, with $\Gamma = \Gamma' \uplus \Gamma''$.

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_u}{\Gamma'x : M \vdash u : A} \lambda \quad \frac{\vdots \mathcal{D}_s}{\Gamma'' \vdash s : M}}{\Gamma' \uplus \Gamma'' \vdash (\lambda x.u)s : A} @$$

By substitution lemma, there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash u\{s/x\} : A$ with $|\mathcal{D}'| = |\mathcal{D}_u| + |\mathcal{D}_s| - |M| < |\mathcal{D}_u| + |\mathcal{D}_s| + 2 = |\mathcal{D}|$. $\square$

Rmk. The quantitative aspect of subject reduction (i.e. $|\mathcal{D}| > |\mathcal{D}'|$) is false:

- if $\mathcal{D}$ is not shrinking, e.g. $\lambda x.x(\delta\delta) \rightarrow_{\ell\beta} \lambda x.x(\delta\delta)$ with $\delta = \lambda z.zz$, see Day 4, p. 10;
- if $t \rightarrow_{\beta} t'$ instead of $t \rightarrow_{\ell\beta} t'$, e.g. $(\lambda z.x)(\delta\delta) \rightarrow_{\beta} (\lambda z.x)(\delta\delta)$ but $(\lambda z.x)(\delta\delta) \rightarrow_{\ell\beta} x$.

Theorem (Correctness of shrinking NI)

If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ is shrinking then there is s β -normal such that $t \rightarrow_{\ell\beta}^k s$ and $|\mathcal{D}| \geq k + |s|$.

Proof. By induction on $|\mathcal{D}|$.

Correctness of NI: shrinking typability implies $\ell\beta$ -normalization

Proposition (Quantitative subject reduction for shrinking derivations)

If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ is shrinking and $t \rightarrow_{\ell\beta} t'$, then there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \rightarrow_{\ell\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \rightarrow_{\ell\beta} u\{s/x\} = t'$: so, $\mathcal{D}$ must have the form below, with $\Gamma = \Gamma' \uplus \Gamma''$.

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_u}{\Gamma'x : M \vdash u : A} \lambda \quad \frac{\vdots \mathcal{D}_s}{\Gamma'' \vdash s : M}}{\Gamma' \vdash \lambda x.u : M \multimap A \quad \Gamma'' \vdash s : M} @$$
$$\frac{}{\Gamma' \uplus \Gamma'' \vdash (\lambda x.u)s : A}$$

By substitution lemma, there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash u\{s/x\} : A$ with $|\mathcal{D}'| = |\mathcal{D}_u| + |\mathcal{D}_s| - |M| < |\mathcal{D}_u| + |\mathcal{D}_s| + 2 = |\mathcal{D}|$. $\square$

Rmk. The **quantitative** aspect of subject reduction (i.e. $|\mathcal{D}| > |\mathcal{D}'|$) is false:

- if $\mathcal{D}$ is not shrinking, e.g. $\lambda x.x(\delta\delta) \rightarrow_{\ell\beta} \lambda x.x(\delta\delta)$ with $\delta = \lambda z.zz$, see Day 4, p. 10;
- if $t \rightarrow_{\beta} t'$ instead of $t \rightarrow_{\ell\beta} t'$, e.g. $(\lambda z.x)(\delta\delta) \rightarrow_{\beta} (\lambda z.x)(\delta\delta)$ but $(\lambda z.x)(\delta\delta) \rightarrow_{\ell\beta} x$.

Theorem (Correctness of shrinking NI)

If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ is shrinking then there is s β -normal such that $t \rightarrow_{\ell\beta}^k s$ and $|\mathcal{D}| \geq k + |s|$.

Proof. By induction on $|\mathcal{D}|$.

Correctness of NI: shrinking typability implies $\ell\beta$ -normalization

Proposition (Quantitative subject reduction for shrinking derivations)

If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ is shrinking and $t \rightarrow_{\ell\beta} t'$, then there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \rightarrow_{\ell\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \rightarrow_{\ell\beta} u\{s/x\} = t'$: so, $\mathcal{D}$ must have the form below, with $\Gamma = \Gamma' \uplus \Gamma''$.

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_u}{\Gamma'x : M \vdash u : A} \lambda \quad \frac{\vdots \mathcal{D}_s}{\Gamma'' \vdash s : M}}{\Gamma' \vdash \lambda x.u : M \multimap A \quad \Gamma'' \vdash s : M} @$$
$$\Gamma' \uplus \Gamma'' \vdash (\lambda x.u)s : A$$

By substitution lemma, there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash u\{s/x\} : A$ with $|\mathcal{D}'| = |\mathcal{D}_u| + |\mathcal{D}_s| - |M| < |\mathcal{D}_u| + |\mathcal{D}_s| + 2 = |\mathcal{D}|$. $\square$

Rmk. The **quantitative** aspect of subject reduction (i.e. $|\mathcal{D}| > |\mathcal{D}'|$) is false:

- if $\mathcal{D}$ is not shrinking, e.g. $\lambda x.x(\delta\delta) \rightarrow_{\ell\beta} \lambda x.x(\delta\delta)$ with $\delta = \lambda z.zz$, see Day 4, p. 10;
- if $t \rightarrow_{\beta} t'$ instead of $t \rightarrow_{\ell\beta} t'$, e.g. $(\lambda z.x)(\delta\delta) \rightarrow_{\beta} (\lambda z.x)(\delta\delta)$ but $(\lambda z.x)(\delta\delta) \rightarrow_{\ell\beta} x$.

Theorem (Correctness of shrinking NI)

If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ is shrinking then there is s β -normal such that $t \rightarrow_{\ell\beta}^k s$ and $|\mathcal{D}| \geq k + |s|$.

Proof. By induction on $|\mathcal{D}|$.

Correctness of NI: shrinking typability implies $\ell\beta$ -normalization

Proposition (Quantitative subject reduction for shrinking derivations)

If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ is shrinking and $t \rightarrow_{\ell\beta} t'$, then there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \rightarrow_{\ell\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \rightarrow_{\ell\beta} u\{s/x\} = t'$: so, $\mathcal{D}$ must have the form below, with $\Gamma = \Gamma' \uplus \Gamma''$.

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_u}{\Gamma'x : M \vdash u : A} \lambda \quad \frac{\vdots \mathcal{D}_s}{\Gamma'' \vdash s : M}}{\Gamma' \vdash \lambda x.u : M \multimap A} \quad \frac{\Gamma' \vdash \lambda x.u : M \multimap A \quad \Gamma'' \vdash s : M}{\Gamma' \uplus \Gamma'' \vdash (\lambda x.u)s : A} @$$

By substitution lemma, there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash u\{s/x\} : A$ with $|\mathcal{D}'| = |\mathcal{D}_u| + |\mathcal{D}_s| - |M| < |\mathcal{D}_u| + |\mathcal{D}_s| + 2 = |\mathcal{D}|$. $\square$

Rmk. The **quantitative** aspect of subject reduction (i.e. $|\mathcal{D}| > |\mathcal{D}'|$) is false:

- if $\mathcal{D}$ is not shrinking, e.g. $\lambda x.x(\delta\delta) \rightarrow_{\ell\beta} \lambda x.x(\delta\delta)$ with $\delta = \lambda z.zz$, see Day 4, p. 10;
- if $t \rightarrow_{\beta} t'$ instead of $t \rightarrow_{\ell\beta} t'$, e.g. $(\lambda z.x)(\delta\delta) \rightarrow_{\beta} (\lambda z.x)(\delta\delta)$ but $(\lambda z.x)(\delta\delta) \rightarrow_{\ell\beta} x$.

Theorem (Correctness of shrinking NI)

If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ is shrinking then there is s β -normal such that $t \rightarrow_{\ell\beta}^k s$ and $|\mathcal{D}| \geq k + |s|$.

Proof. By induction on $|\mathcal{D}|$. If t is β -normal, then the claim follows from the lemma about typing β -normal forms, taking $s = t$ and $k = 0$.

Correctness of NI: shrinking typability implies $\ell\beta$ -normalization

Proposition (Quantitative subject reduction for shrinking derivations)

If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ is shrinking and $t \rightarrow_{\ell\beta} t'$, then there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \rightarrow_{\ell\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \rightarrow_{\ell\beta} u\{s/x\} = t'$: so, $\mathcal{D}$ must have the form below, with $\Gamma = \Gamma' \uplus \Gamma''$.

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_u}{\Gamma'x : M \vdash u : A} \lambda \quad \frac{\vdots \mathcal{D}_s}{\Gamma'' \vdash s : M} \textcircled{\lambda}}{\Gamma' \vdash \lambda x.u : M \multimap A} \quad \frac{\Gamma' \vdash \lambda x.u : M \multimap A \quad \Gamma'' \vdash s : M}{\Gamma' \uplus \Gamma'' \vdash (\lambda x.u)s : A} \textcircled{\lambda}$$

By substitution lemma, there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash u\{s/x\} : A$ with $|\mathcal{D}'| = |\mathcal{D}_u| + |\mathcal{D}_s| - |M| < |\mathcal{D}_u| + |\mathcal{D}_s| + 2 = |\mathcal{D}|$. $\square$

Rmk. The quantitative aspect of subject reduction (i.e. $|\mathcal{D}| > |\mathcal{D}'|$) is false:

- if $\mathcal{D}$ is not shrinking, e.g. $\lambda x.x(\delta\delta) \rightarrow_{\ell\beta} \lambda x.x(\delta\delta)$ with $\delta = \lambda z.zz$, see Day 4, p. 10;
- if $t \rightarrow_{\beta} t'$ instead of $t \rightarrow_{\ell\beta} t'$, e.g. $(\lambda z.x)(\delta\delta) \rightarrow_{\beta} (\lambda z.x)(\delta\delta)$ but $(\lambda z.x)(\delta\delta) \rightarrow_{\ell\beta} x$.

Theorem (Correctness of shrinking NI)

If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ is shrinking then there is s β -normal such that $t \rightarrow_{\ell\beta}^k s$ and $|\mathcal{D}| \geq k + |s|$.

Proof. By induction on $|\mathcal{D}|$. If t is β -normal, then the claim follows from the lemma about typing β -normal forms, taking $s = t$ and $k = 0$.

Otherwise, $t \rightarrow_{\ell\beta} t'$ and by quantitative subject reduction there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| > |\mathcal{D}'|$. By induction hypothesis, $t' \rightarrow_{\ell\beta}^* s$ in k $\ell\beta$ -steps for some β -normal s with $|\mathcal{D}'| \geq k + |s|$. Hence, $t \rightarrow_{\ell\beta}^* s$ in $k+1$ $\ell\beta$ -steps and $|\mathcal{D}| \geq |\mathcal{D}'| + 1 \geq k + 1 + |s|$. $\square$

Ingredients to prove completeness

Rmk. Completeness is the converse of correctness, so their needed ingredients are “dual”.

Lemma (Shrinking typability of β -normal forms)

If t is β -normal, then there is a shrinking $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ for some Γ, A , with $|\mathcal{D}| = |t|$.

Proof. Every β -normal term is of the form $t = \lambda x_n. \dots \lambda x_1. y t_1 \dots t_m$ for some $m, n \in \mathbb{N}$ with $t_1, \dots, t_m$ β -normal. To have the right induction hypothesis, for $n = 0$ we also have to prove that, for all $k \in \mathbb{N}$ and co-shrinking A and shrinking $A_1, \dots, A_n$, there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash y t_1 \dots t_m : [A_1] \multimap \dots \multimap [A_k] \multimap A$ for some co-shrinking environment Γ . The stronger statement is proved by induction on $|t| \in \mathbb{N}$. □

Ingredients to prove completeness

Rmk. Completeness is the converse of correctness, so their needed ingredients are “dual”.

Lemma (Shrinking typability of β -normal forms)

If t is β -normal, then there is a shrinking $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ for some Γ, A , with $|\mathcal{D}| = |t|$.

Proof. Every β -normal term is of the form $t = \lambda x_n. \dots \lambda x_1. y t_1 \dots t_m$ for some $m, n \in \mathbb{N}$ with $t_1, \dots, t_m$ β -normal. To have the right induction hypothesis, for $n = 0$ we also have to prove that, for all $k \in \mathbb{N}$ and co-shrinking A and shrinking $A_1, \dots, A_n$, there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash y t_1 \dots t_m : [A_1] \multimap \dots \multimap [A_k] \multimap A$ for some co-shrinking environment Γ . The stronger statement is proved by induction on $|t| \in \mathbb{N}$. □

Completeness of NI: $\ell\beta$ -normalization implies shrinking typability

Proposition (Quantitative subject expansion for shrinking derivations)

If $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ is shrinking and $t \rightarrow_{\ell\beta} t'$, then there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \rightarrow_{\ell\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \rightarrow_{\ell\beta} u\{s/x\} = t'$: as $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$, by the anti-substitution lemma

there are $\mathcal{D}_u \triangleright_{\text{NI}} \Gamma', x : M \vdash u : A$ and $\mathcal{D}_s \triangleright_{\text{NI}} \Gamma'' \vdash s : M$ such that $\Gamma = \Gamma' \uplus \Gamma''$ and $|\mathcal{D}'| = |\mathcal{D}_u| + |\mathcal{D}_s| - |M|$. Hence, for $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash (\lambda x.u)s : A$ on the left, $|\mathcal{D}| = |\mathcal{D}_u| + |\mathcal{D}_s| + 2 > |\mathcal{D}_u| + |\mathcal{D}_s| - |M| = |\mathcal{D}'|$. $\square$

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_u}{\Gamma'x : M \vdash u : A} \lambda \quad \frac{\vdots \mathcal{D}_s}{\Gamma'' \vdash s : M}}{\Gamma' \uplus \Gamma'' \vdash (\lambda x.u)s : A} @$$

Rmk. We have seen (in day 2) that subject expansion fails with simple types.

Notation. Given $k \in \mathbb{N}$, we write $t \rightarrow_{\ell\beta}^k s$ if $t \xrightarrow[k \text{ } \ell\beta\text{-steps}]{\rightarrow_{\ell\beta} \cdots \rightarrow_{\ell\beta}} s$ (thus $t \rightarrow_{\ell\beta}^0 s$ means $t = s$).

Theorem (Completeness of shrinking NI)

If $t \rightarrow_{\ell\beta}^k s$ with s β -normal, then there is a shrinking $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| \geq k + |s|$.

Proof. By induction on $k \in \mathbb{N}$.

Completeness of NI: $\ell\beta$ -normalization implies shrinking typability

Proposition (Quantitative subject expansion for shrinking derivations)

If $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ is shrinking and $t \rightarrow_{\ell\beta} t'$, then there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \rightarrow_{\ell\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \rightarrow_{\ell\beta} u\{s/x\} = t'$: as $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$, by the anti-substitution lemma

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_u}{\Gamma'x : M \vdash u : A} \lambda \quad \frac{\vdots \mathcal{D}_s}{\Gamma'' \vdash s : M}}{\Gamma' \uplus \Gamma'' \vdash (\lambda x.u)s : A} @$$

there are $\mathcal{D}_u \triangleright_{\text{NI}} \Gamma', x : M \vdash u : A$ and $\mathcal{D}_s \triangleright_{\text{NI}} \Gamma'' \vdash s : M$ such that $\Gamma = \Gamma' \uplus \Gamma''$ and $|\mathcal{D}'| = |\mathcal{D}_u| + |\mathcal{D}_s| - |M|$. Hence, for $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash (\lambda x.u)s : A$ on the left, $|\mathcal{D}| = |\mathcal{D}_u| + |\mathcal{D}_s| + 2 > |\mathcal{D}_u| + |\mathcal{D}_s| - |M| = |\mathcal{D}'|$. $\square$

Rmk. We have seen (in day 2) that subject expansion fails with simple types.

Notation. Given $k \in \mathbb{N}$, we write $t \rightarrow_{\ell\beta}^k s$ if $t \xrightarrow[k \text{ } \ell\beta\text{-steps}]{\rightarrow_{\ell\beta} \cdots \rightarrow_{\ell\beta}} s$ (thus $t \rightarrow_{\ell\beta}^0 s$ means $t = s$).

Theorem (Completeness of shrinking NI)

If $t \rightarrow_{\ell\beta}^k s$ with s β -normal, then there is a shrinking $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| \geq k + |s|$.

Proof. By induction on $k \in \mathbb{N}$.

Completeness of NI: $\ell\beta$ -normalization implies shrinking typability

Proposition (Quantitative subject expansion for shrinking derivations)

If $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ is shrinking and $t \rightarrow_{\ell\beta} t'$, then there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \rightarrow_{\ell\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \rightarrow_{\ell\beta} u\{s/x\} = t'$: as $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$, by the anti-substitution lemma

there are $\mathcal{D}_u \triangleright_{\text{NI}} \Gamma', x : M \vdash u : A$ and $\mathcal{D}_s \triangleright_{\text{NI}} \Gamma'' \vdash s : M$ such that $\Gamma = \Gamma' \uplus \Gamma''$ and $|\mathcal{D}'| = |\mathcal{D}_u| + |\mathcal{D}_s| - |M|$. Hence, for $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash (\lambda x.u)s : A$ on the left, $|\mathcal{D}| = |\mathcal{D}_u| + |\mathcal{D}_s| + 2 > |\mathcal{D}_u| + |\mathcal{D}_s| - |M| = |\mathcal{D}'|$. $\square$

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_u}{\Gamma'x : M \vdash u : A} \lambda \quad \frac{\vdots \mathcal{D}_s}{\Gamma'' \vdash s : M}}{\Gamma' \uplus \Gamma'' \vdash (\lambda x.u)s : A} @$$

Rmk. We have seen (in day 2) that subject expansion fails with simple types.

Notation. Given $k \in \mathbb{N}$, we write $t \rightarrow_{\ell\beta}^k s$ if $t \xrightarrow[k \text{ } \ell\beta\text{-steps}]{\rightarrow_{\ell\beta} \cdots \rightarrow_{\ell\beta}} s$ (thus $t \rightarrow_{\ell\beta}^0 s$ means $t = s$).

Theorem (Completeness of shrinking NI)

If $t \rightarrow_{\ell\beta}^k s$ with s β -normal, then there is a shrinking $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| \geq k + |s|$.

Proof. By induction on $k \in \mathbb{N}$.

Completeness of NI: $\ell\beta$ -normalization implies shrinking typability

Proposition (Quantitative subject expansion for shrinking derivations)

If $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ is shrinking and $t \rightarrow_{\ell\beta} t'$, then there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \rightarrow_{\ell\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \rightarrow_{\ell\beta} u\{s/x\} = t'$: as $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$, by the anti-substitution lemma

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_u}{\Gamma'x : M \vdash u : A} \lambda \quad \frac{\vdots \mathcal{D}_s}{\Gamma'' \vdash s : M}}{\Gamma' \uplus \Gamma'' \vdash (\lambda x.u)s : A} @$$

there are $\mathcal{D}_u \triangleright_{\text{NI}} \Gamma', x : M \vdash u : A$ and $\mathcal{D}_s \triangleright_{\text{NI}} \Gamma'' \vdash s : M$ such that $\Gamma = \Gamma' \uplus \Gamma''$ and $|\mathcal{D}'| = |\mathcal{D}_u| + |\mathcal{D}_s| - |M|$. Hence, for $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash (\lambda x.u)s : A$ on the left, $|\mathcal{D}| = |\mathcal{D}_u| + |\mathcal{D}_s| + 2 > |\mathcal{D}_u| + |\mathcal{D}_s| - |M| = |\mathcal{D}'|$. $\square$

Rmk. We have seen (in day 2) that subject expansion fails with simple types.

Notation. Given $k \in \mathbb{N}$, we write $t \rightarrow_{\ell\beta}^k s$ if $t \xrightarrow[k \text{ } \ell\beta\text{-steps}]{\rightarrow_{\ell\beta} \cdots \rightarrow_{\ell\beta}} s$ (thus $t \rightarrow_{\ell\beta}^0 s$ means $t = s$).

Theorem (Completeness of shrinking NI)

If $t \rightarrow_{\ell\beta}^k s$ with s β -normal, then there is a shrinking $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| \geq k + |s|$.

Proof. By induction on $k \in \mathbb{N}$. If $k = 0$, then $t = s$ and typability of β -normal concludes.

Completeness of NI: $\ell\beta$ -normalization implies shrinking typability

Proposition (Quantitative subject expansion for shrinking derivations)

If $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ is shrinking and $t \rightarrow_{\ell\beta} t'$, then there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \rightarrow_{\ell\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \rightarrow_{\ell\beta} u\{s/x\} = t'$: as $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$, by the anti-substitution lemma

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_u}{\Gamma'x : M \vdash u : A} \lambda \quad \frac{\vdots \mathcal{D}_s}{\Gamma'' \vdash s : M}}{\Gamma' \uplus \Gamma'' \vdash (\lambda x.u)s : A} @$$

there are $\mathcal{D}_u \triangleright_{\text{NI}} \Gamma', x : M \vdash u : A$ and $\mathcal{D}_s \triangleright_{\text{NI}} \Gamma'' \vdash s : M$ such that $\Gamma = \Gamma' \uplus \Gamma''$ and $|\mathcal{D}'| = |\mathcal{D}_u| + |\mathcal{D}_s| - |M|$. Hence, for $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash (\lambda x.u)s : A$ on the left, $|\mathcal{D}| = |\mathcal{D}_u| + |\mathcal{D}_s| + 2 > |\mathcal{D}_u| + |\mathcal{D}_s| - |M| = |\mathcal{D}'|$. $\square$

Rmk. We have seen (in day 2) that subject expansion fails with simple types.

Notation. Given $k \in \mathbb{N}$, we write $t \rightarrow_{\ell\beta}^k s$ if $t \xrightarrow[k \text{ } \ell\beta\text{-steps}]{\rightarrow_{\ell\beta}} \cdots \rightarrow_{\ell\beta} s$ (thus $t \rightarrow_{\ell\beta}^0 s$ means $t = s$).

Theorem (Completeness of shrinking NI)

If $t \rightarrow_{\ell\beta}^k s$ with s β -normal, then there is a shrinking $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| \geq k + |s|$.

Proof. By induction on $k \in \mathbb{N}$. If $k = 0$, then $t = s$ and typability of β -normal concludes. Otherwise $k > 0$ and $t \rightarrow_{\ell\beta} t' \rightarrow_{\ell\beta}^{k-1} s$. By induction hypothesis, there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}'| \geq k - 1 + |s|$. By quantitative subject expansion, there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$, therefore $|\mathcal{D}| \geq |\mathcal{D}'| + 1 \geq k + |s|$. $\square$

Summing up: characterization of leftmost-outermost normalization

Putting together correctness and completeness of NI, we obtain:

Corollary (Characterization of leftmost-outermost normalization)

A term t is $\ell\beta$ -normalizing if and only if there is a shrinking $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$. Moreover, $|\mathcal{D}| \geq k + |s|$ if $t \rightarrow_{\ell\beta}^k s$ with s β -normal.

Rmk. The **quantitative** information about

- the length k of evaluation (left reduction) from t to its β -normal form s , and
- the head size $|s|$ of the β -normal term s

are in the size $|\mathcal{D}|$ of $\mathcal{D}$ **without** performing head reduction $\rightarrow_{\ell\beta}$ or knowing s .

Rmk. $|\mathcal{D}|$ is an **upper bound** to k **plus** $|s|$ together. NI can be refined so that one can:

- 1 **disentangle** the information about k and $|s|$ by means of two different sizes of $\mathcal{D}$,
- 2 obtain the **exact** values of k and $|s|$ from these two sizes of $\mathcal{D}$.

Summing up: characterization of leftmost-outermost normalization

Putting together correctness and completeness of NI, we obtain:

Corollary (Characterization of leftmost-outermost normalization)

A term t is $\ell\beta$ -normalizing if and only if there is a shrinking $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$. Moreover, $|\mathcal{D}| \geq k + |s|$ if $t \rightarrow_{\ell\beta}^k s$ with s β -normal.

Rmk. The **quantitative** information about

- the length k of evaluation (left reduction) from t to its β -normal form s , and
- the head size $|s|$ of the β -normal term s

are in the size $|\mathcal{D}|$ of $\mathcal{D}$ **without** performing head reduction $\rightarrow_{\ell\beta}$ or knowing s .

Rmk. $|\mathcal{D}|$ is an **upper bound** to k **plus** $|s|$ together. NI can be refined so that one can:

- 1 **disentangle** the information about k and $|s|$ by means of two different sizes of $\mathcal{D}$,
- 2 obtain the **exact** values of k and $|s|$ from these two sizes of $\mathcal{D}$.

Summing up: characterization of leftmost-outermost normalization

Putting together correctness and completeness of NI, we obtain:

Corollary (Characterization of leftmost-outermost normalization)

A term t is $\ell\beta$ -normalizing if and only if there is a shrinking $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$. Moreover, $|\mathcal{D}| \geq k + |s|$ if $t \rightarrow_{\ell\beta}^k s$ with s β -normal.

Rmk. The **quantitative** information about

- the length k of evaluation (left reduction) from t to its β -normal form s , and
- the head size $|s|$ of the β -normal term s

are in the size $|\mathcal{D}|$ of $\mathcal{D}$ **without** performing head reduction $\rightarrow_{\ell\beta}$ or knowing s .

Rmk. $|\mathcal{D}|$ is an **upper bound** to k **plus** $|s|$ together. NI can be refined so that one can:

- 1 **disentangle** the information about k and $|s|$ by means of two different sizes of $\mathcal{D}$,
- 2 obtain the **exact** values of k and $|s|$ from these two sizes of $\mathcal{D}$.

Outline

- 1 Characterizing leftmost-outermost normalization in NI
- 2 Some final remarks about non-idempotent intersection types
- 3 Conclusion, exercises and bibliography

The leftmost-outermost reduction is a normalizing strategy

Ex. Let $\delta = \lambda y. yy$, $t = (\lambda z. x)(\delta\delta)$: $t \rightarrow_{\beta} t$ but all derivations of t have the same size.

Proposition (Qualitative subject reduction and expansion)

- 1 If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ and $t \rightarrow_{\beta} t'$, then there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| \geq |\mathcal{D}'|$.
- 2 If $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ and $t \rightarrow_{\beta} t'$, then there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| \geq |\mathcal{D}'|$.

Proof. By induction on the definition of $t \rightarrow_{\beta} t'$ (p. 6 on Day 3). The proof is the same as for $\rightarrow_{h\beta}$ or $\rightarrow_{\ell\beta}$, except that there are more cases, for which $|\mathcal{D}|$ may not decrease. $\square$

Corollary (leftmost-outermost reduction is a normalizing strategy for $\rightarrow_{\beta}$)

If $t \rightarrow_{\beta}^* s$ with s β -normal, then $t \rightarrow_{\ell\beta}^* s$.

Proof. Use shrinking typability of β -normal forms, qualitative subject expansion, correctness for shrinking NI, and confluence of $\rightarrow_{\beta}$. $\square$

Rmk. The corollary is a non-trivial property that has apparently nothing to do with NI.
 $\rightsquigarrow$ NI is **useful** to prove general properties of the untyped λ -calculus.

The leftmost-outermost reduction is a normalizing strategy

Ex. Let $\delta = \lambda y. yy$, $t = (\lambda z. x)(\delta\delta)$: $t \rightarrow_{\beta} t$ but all derivations of t have the same size.

Proposition (Qualitative subject reduction and expansion)

- 1 If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ and $t \rightarrow_{\beta} t'$, then there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| \geq |\mathcal{D}'|$.
- 2 If $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ and $t \rightarrow_{\beta} t'$, then there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| \geq |\mathcal{D}'|$.

Proof. By induction on the definition of $t \rightarrow_{\beta} t'$ (p. 6 on Day 3). The proof is the same as for $\rightarrow_{h\beta}$ or $\rightarrow_{\ell\beta}$, except that there are more cases, for which $|\mathcal{D}|$ may not decrease. $\square$

Corollary (leftmost-outermost reduction is a normalizing strategy for $\rightarrow_{\beta}$)

If $t \rightarrow_{\beta}^* s$ with s β -normal, then $t \rightarrow_{\ell\beta}^* s$.

Proof. Use shrinking typability of β -normal forms, qualitative subject expansion, correctness for shrinking NI, and confluence of $\rightarrow_{\beta}$. $\square$

Rmk. The corollary is a non-trivial property that has apparently nothing to do with NI.
 $\rightsquigarrow$ NI is **useful** to prove general properties of the untyped λ -calculus.

The leftmost-outermost reduction is a normalizing strategy

Ex. Let $\delta = \lambda y. yy$, $t = (\lambda z. x)(\delta\delta)$: $t \rightarrow_{\beta} t$ but all derivations of t have the same size.

Proposition (Qualitative subject reduction and expansion)

- 1 If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ and $t \rightarrow_{\beta} t'$, then there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| \geq |\mathcal{D}'|$.
- 2 If $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ and $t \rightarrow_{\beta} t'$, then there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| \geq |\mathcal{D}'|$.

Proof. By induction on the definition of $t \rightarrow_{\beta} t'$ (p. 6 on Day 3). The proof is the same as for $\rightarrow_{h\beta}$ or $\rightarrow_{\ell\beta}$, except that there are more cases, for which $|\mathcal{D}|$ may not decrease. $\square$

Corollary (leftmost-outermost reduction is a normalizing strategy for $\rightarrow_{\beta}$)

If $t \rightarrow_{\beta}^* s$ with s β -normal, then $t \rightarrow_{\ell\beta}^* s$.

Proof. Use shrinking typability of β -normal forms, qualitative subject expansion, correctness for shrinking NI, and confluence of $\rightarrow_{\beta}$. $\square$

Rmk. The corollary is a non-trivial property that has apparently nothing to do with NI.
 $\rightsquigarrow$ NI is **useful** to prove general properties of the untyped λ -calculus.

The leftmost-outermost reduction is a normalizing strategy

Ex. Let $\delta = \lambda y. yy$, $t = (\lambda z. x)(\delta\delta)$: $t \rightarrow_{\beta} t$ but all derivations of t have the same size.

Proposition (Qualitative subject reduction and expansion)

- 1 If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ and $t \rightarrow_{\beta} t'$, then there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| \geq |\mathcal{D}'|$.
- 2 If $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ and $t \rightarrow_{\beta} t'$, then there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| \geq |\mathcal{D}'|$.

Proof. By induction on the definition of $t \rightarrow_{\beta} t'$ (p. 6 on Day 3). The proof is the same as for $\rightarrow_{h\beta}$ or $\rightarrow_{\ell\beta}$, except that there are more cases, for which $|\mathcal{D}|$ may not decrease. $\square$

Corollary (leftmost-outermost reduction is a normalizing strategy for $\rightarrow_{\beta}$)

If $t \rightarrow_{\beta}^* s$ with s β -normal, then $t \rightarrow_{\ell\beta}^* s$.

Proof. Use shrinking typability of β -normal forms, qualitative subject expansion, correctness for shrinking NI, and confluence of $\rightarrow_{\beta}$. $\square$

Rmk. The corollary is a non-trivial property that has apparently nothing to do with NI.
 $\rightsquigarrow$ NI is **useful** to prove general properties of the untyped λ -calculus.

The philosophy behind the proofs characterizing normalizations via NI

Our proofs to characterize normalization for $\rightarrow_r$ ($r \in \{h\beta, \ell\beta\}$) follow the same pattern:

normalizability for $\rightarrow_r$	$\Rightarrow$	typability in NI (for some types)	$\Rightarrow$	strong normalizability for $\rightarrow_r$
--	---------------	--------------------------------------	---------------	---

Indeed, correctness is proved via quantitative subject reduction (p. 15 Day 4/p. 8 Day 5): after **every** single r -step from a typable term, the size of the derivation decreases.

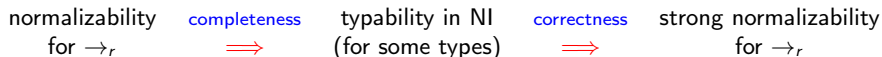
$\rightsquigarrow$ This is a way to prove that **normalization** and **strong normalization** coincide for $\rightarrow_r$ (t is r -normalizing iff t is strongly r -normalizing, see definitions on p. 11 of Day 3).

$\rightarrow_{h\beta}/\rightarrow_{\ell\beta}$ is deterministic $\rightsquigarrow$ trivially normaliz. and strong normaliz. coincide for it.

Nevertheless, *a priori*, the NI method could be used to prove that normalization and strong normalization coincide for a non-deterministic reduction.

The philosophy behind the proofs characterizing normalizations via NI

Our proofs to characterize normalization for $\rightarrow_r$ ($r \in \{h\beta, \ell\beta\}$) follow the same pattern:



Indeed, correctness is proved via quantitative subject reduction (p. 15 Day 4/p. 8 Day 5): after **every** single r -step from a typable term, the size of the derivation decreases.

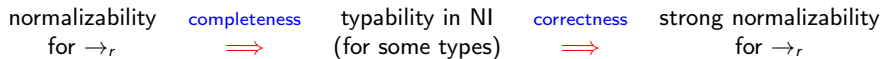
$\rightsquigarrow$ This is a way to prove that **normalization** and **strong normalization** coincide for $\rightarrow_r$ (t is r -normalizing iff t is strongly r -normalizing, see definitions on p. 11 of Day 3).

$\rightarrow_{h\beta}/\rightarrow_{\ell\beta}$ is deterministic $\rightsquigarrow$ trivially normaliz. and strong normaliz. coincide for it.

Nevertheless, *a priori*, the NI method could be used to prove that normalization and strong normalization coincide for a non-deterministic reduction.

The philosophy behind the proofs characterizing normalizations via NI

Our proofs to characterize normalization for $\rightarrow_r$ ($r \in \{h\beta, \ell\beta\}$) follow the same pattern:



Indeed, correctness is proved via quantitative subject reduction (p. 15 Day 4/p. 8 Day 5): after **every** single r -step from a typable term, the size of the derivation decreases.

$\rightsquigarrow$ This is a way to prove that **normalization** and **strong normalization** coincide for $\rightarrow_r$ (t is r -normalizing iff t is strongly r -normalizing, see definitions on p. 11 of Day 3).

$\rightarrow_{h\beta}/\rightarrow_{\ell\beta}$ is deterministic $\rightsquigarrow$ trivially normaliz. and strong normaliz. coincide for it.

Nevertheless, *a priori*, the NI method could be used to prove that normalization and strong normalization coincide for a non-deterministic reduction.

Outline

- 1 Characterizing leftmost-outermost normalization in NI
- 2 Some final remarks about non-idempotent intersection types
- 3 Conclusion, exercises and bibliography

What we have learned today?

- ➊ How to use the non-idempotent intersection type system NI with specific types.
- ➋ Characterization of leftmost-outermost normalization via NI.
- ➌ A combinatorial proof for that characterization.
- ➍ How to extract quantitative information from shrinking derivations in NI.
- ➎ A proof that leftmost-outermost is a normalizing strategy, based on NI.

Questions?

What we have learned today?

- ➊ How to use the non-idempotent intersection type system NI with specific types.
- ➋ Characterization of leftmost-outermost normalization via NI.
- ➌ A combinatorial proof for that characterization.
- ➍ How to extract quantitative information from shrinking derivations in NI.
- ➎ A proof that leftmost-outermost is a normalizing strategy, based on NI.

Questions?

What we have learned today?

- ➊ How to use the non-idempotent intersection type system NI with specific types.
- ➋ Characterization of leftmost-outermost normalization via NI.
- ➌ A combinatorial proof for that characterization.
- ➍ How to extract quantitative information from shrinking derivations in NI.
- ➎ A proof that leftmost-outermost is a normalizing strategy, based on NI.

Questions?

What we have learned today?

- ➊ How to use the non-idempotent intersection type system NI with specific types.
- ➋ Characterization of leftmost-outermost normalization via NI.
- ➌ A combinatorial proof for that characterization.
- ➍ How to extract quantitative information from shrinking derivations in NI.
- ➎ A proof that leftmost-outermost is a normalizing strategy, based on NI.

Questions?

What we have learned today?

- 1 How to use the non-idempotent intersection type system NI with specific types.
- 2 Characterization of leftmost-outermost normalization via NI.
- 3 A combinatorial proof for that characterization.
- 4 How to extract quantitative information from shrinking derivations in NI.
- 5 A proof that leftmost-outermost is a normalizing strategy, based on NI.

Questions?



Exercises

- ❶ Find all shrinking derivations of $x : M \vdash xx : C$, for any linear C and any multi M .
- ❷ Find all shrinking derivations of $x : M, y : N \vdash xy : C$, for any linear C and multi M, N .
- ❸ Find all derivations of $\vdash (\lambda x.xx)\lambda y.y : A$ for any shrinking linear type A .
- ❹ Prove that there is no derivation of $\vdash \lambda x.x(\delta\delta) : C$ for any shrinking linear type C .
- ❺ Prove that all derivations of $\vdash (\lambda z.x)(\delta\delta) : C$ for any shrinking linear type C have the same size. Deduce that quantitative subject reduction on shrinking derivations (proposition on p. 7) does not hold in general when $t \rightarrow_{\beta} t'$ instead of $t \rightarrow_{\ell\beta} t'$.
- ❻ Prove rigorously the two lemmas on p. 7 and the lemma on p. 9.
- ❼ Prove rigorously the quantitative subject reduction (p. 8) and expansion (p. 10), by induction on the definition of $t \rightarrow_{\ell\beta} t'$ (see Day 3, p. 9).
- ❽ Prove rigorously the qualitative versions of subject reduction and expansion (Proposition on p. 12).
- ❾ Prove rigorously that $\rightarrow_{\ell\beta}$ is a normalizing strategy for $\rightarrow_{\beta}$ (Corollary on p. 12).
- ❿ Do we really need quantitative subject expansion for shrinking derivations (proposition on p. 10) to prove completeness of shrinking NI (theorem on p. 10)?
Hint: Use the qualitative version of subject expansion (Proposition on p. 12).

Bibliography

- For an (almost gentle) introduction to non-idempotent intersection types:



Antonio Bucciarelli, Delia Kesner, Daniel Ventura. *Non-Idempotent Intersection types for the Lambda-Calculus*. Logic Journal of the IGPL, vol. 25, issue 4, pp. 431–464, 2017. <https://doi.org/10.1093/jigpal/jzx018>

- For a very advanced study about non-idempotent intersection types:



Beniamino Accattoli, Stéphan Graham-Lengrand, Delia Kesner. *Tight typings and split bounds, fully developed*. Journal of Functional Programming, vol. 30, 14 pages, 2020. <https://doi.org/10.1017/S095679682000012X>