

The λ -calculus: from simple types to non-idempotent intersection types

Day 4: Non-idempotent intersection types for the λ -calculus

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Outline

- 1 Non-idempotent intersection types for the λ -calculus
- 2 Characterizing head normalization in NI
- 3 Conclusion, exercises and bibliography

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The λ -calculus between simple types and the untyped one

The **simply typed** λ -calculus:

- ➊ has very nice operational properties (e.g. normalization, confluence);
- ➋ has a clear logical meaning (Curry-Howard correspondence);
- ➌ is not very expressive (recursion cannot be represented, Turing-completeness fails).

The **untyped** λ -calculus:

- ➊ has some very nice properties (e.g. confluence, Turing-completeness);
- ➋ misses some nice properties (e.g. normalization);
- ➌ has no logical meaning;
- ➍ contains diverging terms without any meaning (e.g. $\delta\delta$).

Questions.

- ➊ Is there a **more liberal** type system which only takes the pros of the two worlds?
- ➋ Can it characterize all and only the “**meaningful**” terms of the untyped λ -calculus?

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The syntax for non-idempotent intersection types

We fix a countably infinite set of **atoms**, denoted by $X, Y, Z, \dots$

Linear types: $A, B ::= X \mid M \multimap A$

Multi types: $M, N ::= [A_1, \dots, A_n]$ (with $n \in \mathbb{N}$)

(Non-idempotent intersection) types: $S, T ::= A \mid M$

where $[A_1, \dots, A_n]$ with $n \in \mathbb{N}$ is a finite multiset ($[]$ is the empty multiset for $n = 0$).

Idea. $[A_1, \dots, A_n]$ stands for a conjunction $A_1 \wedge \dots \wedge A_n$ where $\wedge$ is:

- **commutative** $A \wedge B \equiv B \wedge A$ (multisets do not take order into account);
- **associative** $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$ (multisets are associative);
- **non-idempotent** $A \wedge A \not\equiv A$ (multisets take multiplicities into account).

Def. A **judgment** is a sequent of the form $\Gamma \vdash t : T$ where

- 1 t is a term, T is a type, Γ is an **environment**, that is,
- 2 Γ is a function from variables to multi types such that $\{x \mid \Gamma(x) \neq []\}$ is a finite set.

Notation. $\uplus$ is the multiset union (e.g. $[A, B] \uplus [A] = [A, A, B] \neq [A, B]$) whose unit is $[]$.
Extended to type environments pointwise: $(\Gamma \uplus \Delta)(x) = \Gamma(x) \uplus \Delta(x)$.

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The typing rules for non-idempotent intersection type system NI

Notation. An environment Γ is denoted by $x_1 : M_1, \dots, x_n : M_n$ if:

variables $x_1, \dots, x_n$ are pairwise distinct and $\Gamma(x) = \begin{cases} M_i & \text{if } x = x_i \text{ for some } 1 \leq i \leq n, \\ [] & \text{otherwise.} \end{cases}$

Typing rules for NI:

$$\frac{}{x : [A] \vdash x : A} \text{var}$$

$$\frac{\Gamma, x : M \vdash t : A}{\Gamma \vdash \lambda x. t : M \multimap A} \lambda$$

$$\frac{\Gamma \vdash s : M \multimap A \quad \Delta \vdash t : M}{\Gamma \uplus \Delta \vdash st : A} @$$

$$\frac{(\Gamma_i \vdash t : A_i)_{1 \leq i \leq n} \quad n \in \mathbb{N}}{\biguplus_{i=1}^n \Gamma_i \vdash t : [A_1, \dots, A_n]} !$$

Idea. A term typed $t : [A, A, B]$ means that, during evaluation, t can be used:

- once as a data of type B , and
- twice as a data of type A .

Notation. $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : T$ means that $\mathcal{D}$ is a derivation in NI with conclusion $\Gamma \vdash t : T$.
 $\Gamma \vdash_{\text{NI}} t : T$ means that there is a derivation $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : T$.

Rmk. $\vdash_{\text{NI}} t : []$ for every term t (take ! with no premises).

Def. The size $|\mathcal{D}|$ of a derivation $\mathcal{D}$ is the number of its rules, not counting the rules !.
 $|\mathcal{D}|_{\text{var}}$ (resp. $|\mathcal{D}|_{\lambda}$; $|\mathcal{D}|_{@}$) is the number of rules var (resp. λ ; $@$) in $\mathcal{D}$.

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Some examples of derivations in NI

Ex. Find all the derivations with conclusion $\vdash \lambda x.x : C$, for any linear type C .

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Some examples of derivations in NI

Ex. Find all the derivations with conclusion $\vdash \lambda x.x : C$, for any linear type C .

$$\mathcal{D}'_A = \frac{\overline{x : [A] \vdash x : A}^{\text{var}}}{\vdash \lambda x.x : [A] \multimap A} \lambda \quad \text{for any linear type } A.$$

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Ex. Find all the derivations with conclusion $\vdash \lambda x.xx : C$, for any linear type C .

$$\mathcal{D}_{A_0, \dots, A_n}^{\delta, n} = \frac{\overline{x : [[A_1, \dots, A_n] \multimap A_0] \vdash x : [A_1, \dots, A_n] \multimap A_0}^{\text{var}} \quad \frac{\left(\overline{x : [A_i] \vdash x : A_i}^{\text{var}} \right)_{1 \leq i \leq n} !}{x : [A_1, \dots, A_n] \vdash x : [A_1, \dots, A_n]} \circledast}{\frac{x : [[A_1, \dots, A_n] \multimap A_0, A_1, \dots, A_n] \vdash xx : A_0}{\vdash \lambda x.xx : [[A_1, \dots, A_n] \multimap A_0, A_1, \dots, A_n] \multimap A_0} \lambda} \lambda$$

for any $n \in \mathbb{N}$ and any linear types $A_0, \dots, A_n$ (in particular, for $n = 0$, $\vdash \lambda x.x : [[] \multimap A_0] \multimap A_0$).

More examples of derivations in NI

Ex. Find all the derivations with conclusion $\vdash (\lambda x.x)\lambda y.y : C$, for any linear type C .

Ex. Find a derivation with conclusion $\vdash (\lambda x.xx)\lambda y.y : C$, for some linear type C .

Rmk. In the derivation $\mathcal{D}_A^{II}$ (resp. $\mathcal{D}_A^{\delta, I}$) the rule ! has 1 premise (resp. 2 premises) because 1 copy (resp. 2 copies) of $\lambda y.y$ is (resp. are) needed in the evaluation $(\lambda x.x)\lambda y.y \rightarrow_{h\beta} \lambda y.y$ (resp. $(\lambda x.xx)\lambda y.y \rightarrow_{h\beta} (\lambda y.y)\lambda y.y$).

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$$\mathcal{D}_A'' = \frac{\frac{\frac{}{x : [[A] \multimap A] \vdash x : [A] \multimap A} \text{var}}{\vdash \lambda x.x : [[A] \multimap A] \multimap [A] \multimap A} \lambda \quad \frac{\frac{\frac{}{y : [A] \vdash y : A} \text{var}}{\vdash \lambda y.y : [A] \multimap A} \lambda}{\vdash \lambda y.y : [[A] \multimap A]} !}{\vdash (\lambda x.x)\lambda y.y : [A] \multimap A} \textcircled{!}$$

for any linear type A .

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Ex. Find a derivation with conclusion $\vdash (\lambda x.xx)\lambda y.y : C$, for some linear type C .

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for any linear type A .

Ex. Find a derivation with conclusion $\vdash (\lambda x.xx)\lambda y.y : C$, for some linear type C .

$$\mathcal{D}_A^{\delta, I} = \frac{\vdash \lambda x.xx : [[[A] \multimap A] \multimap [A] \multimap A, [A] \multimap A] \multimap [A] \multimap A \quad \frac{\vdash \lambda y.y : [[A] \multimap A] \multimap [A] \multimap A \quad \vdash \lambda y.y : [A] \multimap A}{\vdash \lambda y.y : [[[A] \multimap A] \multimap [A] \multimap A, [A] \multimap A]} !}{\vdash (\lambda x.xx)\lambda y.y : [A] \multimap A} @$$

for any linear type A (actually, all derivations for $(\lambda x.xx)\lambda y.y$ have the form above).

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Oh no! More examples of derivations in NI!

Ex. Find a derivation with conclusion $\vdash (\lambda x.xx)((\lambda y.y)\lambda z.z) : C$, for some linear type C .

Rmk. In the derivation $\mathcal{D}_A^{\delta, II}$, the rule $!$ has 2 premises because 2 copies of $(\lambda y.y)\lambda z.z$ are needed in the evaluation $(\lambda x.xx)((\lambda y.y)\lambda z.z) \rightarrow_{h\beta} ((\lambda y.y)\lambda z.z)((\lambda y.y)\lambda z.z)$.

In turn, in each of the derivations $\mathcal{D}_{[[A] \multimap A] \multimap [A] \multimap A}^{II}$ and $\mathcal{D}_{[A] \multimap A}^{II}$ the rule $!$ has 2 premises, hence the derivation $\mathcal{D}_A^{\delta, II}$ has 4 subderivations with conclusion $\lambda x.x$, because 4 copies of $\lambda x.x$ are needed in the evaluation $(\lambda x.xx)((\lambda y.y)\lambda z.z) \rightarrow_{h\beta} ((\lambda y.y)\lambda z.z)((\lambda y.y)\lambda z.z)$.

Ex. Find a derivation with conclusion $\vdash (\lambda x.xx)\lambda y.yy : C$, for some linear type C .

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Ex. Find a derivation with conclusion $\vdash (\lambda x.xx)((\lambda y.y)\lambda z.z) : C$, for some linear type C .

$$\mathcal{D}_A^{\delta, II} = \frac{\frac{\vdots \mathcal{D}_{[A] \multimap A, [A] \multimap A}^{\delta, \mathbf{1}}}{\vdash \lambda x.xx : [[A] \multimap A] \multimap [A] \multimap A, [A] \multimap A} \quad \frac{\vdots \mathcal{D}_{[[A] \multimap A] \multimap [A] \multimap A}^{II} \quad \vdots \mathcal{D}_{[A] \multimap A}^{II}}{\vdash (\lambda y.y)\lambda z.z : [[A] \multimap A] \multimap [A] \multimap A \quad \vdash (\lambda y.y)\lambda z.z : [A] \multimap A}}{\vdash (\lambda y.y)\lambda z.z : [[[A] \multimap A] \multimap [A] \multimap A, [A] \multimap A]} \quad \frac{\vdash \lambda x.xx : [[A] \multimap A] \multimap [A] \multimap A, [A] \multimap A \quad \vdash (\lambda y.y)\lambda z.z : [[[A] \multimap A] \multimap [A] \multimap A, [A] \multimap A]}{\vdash (\lambda x.xx)((\lambda y.y)\lambda z.z) : [A] \multimap A} ! @$$

for any linear type A (actually, all derivations for $(\lambda x.xx)((\lambda y.y)\lambda z.z)$ have that form).

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for any linear type A (actually, all derivations for $(\lambda x.xx)((\lambda y.y)\lambda z.z)$ have that form).

Rmk. In the derivation $\mathcal{D}_A^{\delta, II}$, the rule ! has 2 premises because 2 copies of $(\lambda y.y)\lambda z.z$ are needed in the evaluation $(\lambda x.xx)((\lambda y.y)\lambda z.z) \rightarrow_{h\beta} ((\lambda y.y)\lambda z.z)((\lambda y.y)\lambda z.z)$.

In turn, in each of the derivations $\mathcal{D}_{[[A] \multimap A] \multimap [A] \multimap A}^{II}$ and $\mathcal{D}_{[A] \multimap A}^{II}$ the rule ! has 2 premises, hence the derivation $\mathcal{D}_A^{\delta, II}$ has 4 subderivations with conclusion $\lambda x.x$, because 4 copies of $\lambda x.x$ are needed in the evaluation $(\lambda x.xx)((\lambda y.y)\lambda z.z) \rightarrow_{h\beta} ((\lambda y.y)\lambda z.z)((\lambda y.y)\lambda z.z)$.

Ex. Find a derivation with conclusion $\vdash (\lambda x.xx)\lambda y.yy : C$, for some linear type C .

Oh no! More examples of derivations in NI!

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Good luck!

Enough with the examples of derivations, old man!

Ex. Find all derivations with conclusion $\vdash \lambda x.x((\lambda y.yy)\lambda z.zz):C$, for any linear type C .

Ex. Find a derivation for $F = \lambda a.\lambda f.f(aaf)$ and one for $\Theta = FF$ with some linear type.

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This is a good exercise, old man!

Outline

- 1 Non-idempotent intersection types for the λ -calculus
- 2 Characterizing head normalization in NI
- 3 Conclusion, exercises and bibliography

What can we do with non-idempotent intersection types?

Goal. We want to characterize all and only the $h\beta$ -normalizing terms via NI.

Motivation. There are many theoretical reasons to say “meaningful” = $h\beta$ -normalizing.

To achieve this qualitative characterization, we need to prove two properties.

- ① **Correctness:** if a term is typable in NI then it is $h\beta$ -normalizing.
- ② **Completeness:** if a term is $h\beta$ -normalizing then it is typable in NI.

Bonus. We can extract some quantitative information from NI about:

- ① the **length of evaluation** (the number of $h\beta$ -steps to reach the $h\beta$ -normal form);
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Ingredients to prove correctness

Def. The **head size** $|t|_{h\beta}$ of a term t is defined by induction on t as follows:

$$|x|_{h\beta} = 1 \qquad |\lambda x.t|_{h\beta} = 1 + |t|_{h\beta} \qquad |st|_{h\beta} = 1 + |s|_{h\beta}$$

Lemma (Typing $h\beta$ -normal forms)

Let t be $h\beta$ -normal. If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ then $|t|_{h\beta} \leq |\mathcal{D}|$.

Proof. Every $h\beta$ -normal term is of the form $t = \lambda x_n \dots \lambda x_1. y t_1 \dots t_m$ for some $m, n \in \mathbb{N}$. The lemma is proved by induction on $|t|_{h\beta} \in \mathbb{N}$. $\square$

Notation. For a finite multiset M over a set X , its **cardinality** is $|M| = \sum_{x \in X} M(x) \in \mathbb{N}$.

Lemma (Substitution)

If $\mathcal{D} \triangleright_{\text{NI}} \Gamma, x : M \vdash t : A$ and $\mathcal{D}' \triangleright_{\text{NI}} \Delta \vdash s : M$, then there is $\mathcal{D}'' \triangleright_{\text{NI}} \Gamma \uplus \Delta \vdash t\{s/x\} : A$ with $|\mathcal{D}''| = |\mathcal{D}| + |\mathcal{D}'| - |M|$.

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A graphical view to the substitution lemma

Like natural deduction, derivations in NI can be depicted by a tree-like structure where:

- edges are labeled by typed terms, nodes are the typing rules,
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$$\begin{array}{c} \vdots \mathcal{D} \\ x_1 : [A_{11}, \dots A_{1k_1}], \dots, x_n : [A_{n1}, \dots A_{nk_n}] \vdash t : T \end{array}$$

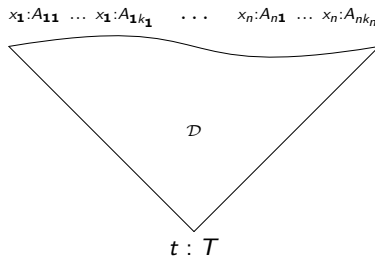
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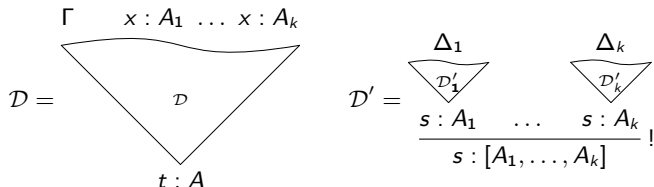
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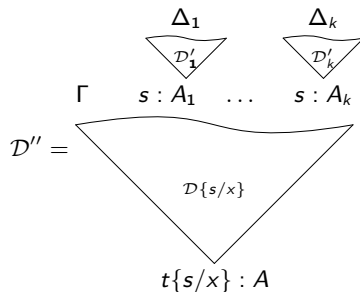
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Correctness of NI: typability implies $h\beta$ -normalization

Proposition (Quantitative subject reduction)

If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ and $t \rightarrow_{h\beta} t'$, then there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \rightarrow_{h\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \rightarrow_{h\beta} u\{s/x\} = t'$: so, $\mathcal{D}$ must have the form below, with $\Gamma = \Gamma' \uplus \Gamma''$.

$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_u}{\Gamma'x : M \vdash u : A} \lambda \quad \frac{\vdots \mathcal{D}_s}{\Gamma'' \vdash s : M}}{\Gamma' \vdash \lambda x.u : M \multimap A} \quad \frac{\Gamma' \vdash \lambda x.u : M \multimap A \quad \Gamma'' \vdash s : M}{\Gamma' \uplus \Gamma'' \vdash (\lambda x.u)s : A} @$$

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Rmk. The **quantitative** aspect of subject reduction (i.e. $|\mathcal{D}| > |\mathcal{D}'|$) is false:

- if $t \rightarrow_{\beta} t'$ instead of $t \rightarrow_{h\beta} t'$, e.g. $\lambda x.x(\delta\delta) \rightarrow_{\beta} \lambda x.x(\delta\delta)$ with $\delta = \lambda z.zz$, see p. 10;
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Theorem (Correctness of NI)

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$$\mathcal{D} = \frac{\frac{\vdots \mathcal{D}_u}{\Gamma'x : M \vdash u : A} \lambda \quad \frac{\vdots \mathcal{D}_s}{\Gamma'' \vdash s : M}}{\Gamma' \vdash \lambda x.u : M \multimap A} \quad \frac{\Gamma' \vdash \lambda x.u : M \multimap A \quad \Gamma'' \vdash s : M}{\Gamma' \uplus \Gamma'' \vdash (\lambda x.u)s : A} @$$

By substitution lemma, there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash u\{s/x\} : A$ with $|\mathcal{D}'| = |\mathcal{D}_u| + |\mathcal{D}_s| - |M| < |\mathcal{D}_u| + |\mathcal{D}_s| + 2 = |\mathcal{D}|$. $\square$

Rmk. The **quantitative** aspect of subject reduction (i.e. $|\mathcal{D}| > |\mathcal{D}'|$) is false:

- if $t \rightarrow_{\beta} t'$ instead of $t \rightarrow_{h\beta} t'$, e.g. $\lambda x.x(\delta\delta) \rightarrow_{\beta} \lambda x.x(\delta\delta)$ with $\delta = \lambda z.zz$, see p. 10;
- if $\mathcal{D}$ and $\mathcal{D}'$ are derivations in the simply typed λ -calculus, instead of NI.

Theorem (Correctness of NI)

If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ then there is s $h\beta$ -normal such that $t \xrightarrow[k \text{ } h\beta\text{-steps}]{\rightarrow_{h\beta} \cdots \rightarrow_{h\beta}} s$ and $|\mathcal{D}| \geq k + |s|_{h\beta}$.

Proof. By induction on $|\mathcal{D}|$. If t is $h\beta$ -normal, then the claim follows from the lemma about typing $h\beta$ -normal forms, taking $s = t$ and $k = 0$.

Correctness of NI: typability implies $h\beta$ -normalization

Proposition (Quantitative subject reduction)

If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ and $t \rightarrow_{h\beta} t'$, then there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \rightarrow_{h\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \rightarrow_{h\beta} u\{s/x\} = t'$: so, $\mathcal{D}$ must have the form below, with $\Gamma = \Gamma' \uplus \Gamma''$.

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By substitution lemma, there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash u\{s/x\} : A$ with $|\mathcal{D}'| = |\mathcal{D}_u| + |\mathcal{D}_s| - |M| < |\mathcal{D}_u| + |\mathcal{D}_s| + 2 = |\mathcal{D}|$. $\square$

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Proof. By induction on $|\mathcal{D}|$. If t is $h\beta$ -normal, then the claim follows from the lemma about typing $h\beta$ -normal forms, taking $s = t$ and $k = 0$.

Otherwise, $t \rightarrow_{h\beta} t'$ and by quantitative subject reduction there is $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| > |\mathcal{D}'|$. By induction hypothesis, $t' \rightarrow_{h\beta}^* s$ in k $h\beta$ -steps for some $h\beta$ -normal s with $|\mathcal{D}'| \geq k + |s|_{h\beta}$. Hence, $t \rightarrow_{h\beta}^* s$ in $k+1$ $h\beta$ -steps and $|\mathcal{D}| \geq |\mathcal{D}'| + 1 \geq k + 1 + |s|_{h\beta}$. $\square$

Ingredients to prove completeness

Rmk. Completeness is the converse of correctness, so their needed ingredients are “dual”.

Lemma (Typability of $h\beta$ -normal forms)

If t is $h\beta$ -normal, then there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| = |t|_{h\beta}$.

Proof. Every $h\beta$ -normal term is of the form $t = \lambda x_n \dots \lambda x_1. y t_1 \dots t_m$ for some $m, n \in \mathbb{N}$. For $n = 0$, we prove by induction on $m \in \mathbb{N}$ the stronger property that, for all $k \in \mathbb{N}$ and linear A , there is $\mathcal{D} \triangleright_{\text{NI}} y : [A_k] \vdash y t_1 \dots t_m : A_k$ with $|\mathcal{D}| = m + 1 = |\mathcal{D}|_{@} + |\mathcal{D}|_{\text{var}}$ and

$$A_k = \underbrace{[] \multimap \dots \multimap []}_{k \text{ times } []} \multimap A \quad (\text{note that } |y t_1 \dots t_m|_{h\beta} = m + 1 \text{ and } |\mathcal{D}|_{@} = m).$$

The lemma including the stronger statement is proved by induction on $|t|_{h\beta} \in \mathbb{N}$. □

Lemma (Anti-substitution)

If $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t\{s/u\} : A$, then there are environments Γ' and Γ'' , a multi type M and derivations $\mathcal{D}' \triangleright_{\text{NI}} \Gamma', x : M \vdash t : A$ and $\mathcal{D}'' \triangleright_{\text{NI}} \Gamma'' \vdash s : M$ such that $\Gamma = \Gamma' \uplus \Gamma''$ and $|\mathcal{D}| = |\mathcal{D}'| + |\mathcal{D}''| - |M|$.

Proof. By structural induction on t . The base case is when t is a variable (either x or other than x). The other cases follow easily from the inductive hypothesis. □

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Completeness of NI: $h\beta$ -normalization implies typability

Proposition (Quantitative subject expansion)

If $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ and $t \rightarrow_{h\beta} t'$, then there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \rightarrow_{h\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \rightarrow_{h\beta} u\{s/x\} = t'$: as $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$, by the anti-substitution lemma

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Rmk. We have seen (in day 2) that subject expansion fails with simple types.

Notation. Given $k \in \mathbb{N}$, we write $t \rightarrow_{h\beta}^k s$ if $t \xrightarrow[k \text{ } h\beta\text{-steps}]{\rightarrow_{h\beta} \cdots \rightarrow_{h\beta}} s$ (thus $t \rightarrow_{h\beta}^0 s$ means $t = s$).

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If $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ and $t \rightarrow_{h\beta} t'$, then there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

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Theorem (Completeness of NI)

If $t \rightarrow_{h\beta}^k s$ with s $h\beta$ -normal, then there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| \geq k + |s|_{h\beta}$.

Proof. By induction on $k \in \mathbb{N}$. If $k = 0$, then $t = s$ and typability of $h\beta$ -normal concludes.

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If $\mathcal{D}' \triangleright_{\text{NI}} \Gamma \vdash t' : A$ and $t \rightarrow_{h\beta} t'$, then there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

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Summing up: characterization of head normalization

Putting together correctness and completeness of NI, we obtain:

Corollary (Characterization of head normalization)

A term t is $h\beta$ -normalizing if and only if there is $\mathcal{D} \triangleright_{\text{NI}} \Gamma \vdash t : A$. Moreover, $|\mathcal{D}| \geq k + |s|_{h\beta}$ if $t \rightarrow_{h\beta}^k s$ with s $h\beta$ -normal.

Rmk. The **quantitative** information about

- the length k of evaluation (head reduction) from t to its $h\beta$ -normal form s , and
- the head size $|s|_{h\beta}$ of the $h\beta$ -normal term s

are in the size $|\mathcal{D}|$ of $\mathcal{D}$ **without** performing head reduction $\rightarrow_{h\beta}$ or knowing s .

Rmk. $|\mathcal{D}|$ is an **upper bound** to k **plus** $|s|_{h\beta}$ together. NI can be refined so that one can:

- 1 **disentangle** the information about k and $|s|_{h\beta}$ by means of two different sizes of $\mathcal{D}$,
- 2 obtain the **exact** values of k and $|s|_{h\beta}$ from these two sizes of $\mathcal{D}$.

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Outline

- 1 Non-idempotent intersection types for the λ -calculus
- 2 Characterizing head normalization in NI
- 3 Conclusion, exercises and bibliography

What we have learned today?

- 1 The non-idempotent intersection type system NI.
- 2 Characterization of head normalization via NI.
- 3 A combinatorial proof for that characterization.
- 4 How to extract quantitative information from derivations in NI.

Questions?

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- 1 The non-idempotent intersection type system NI .
- 2 Characterization of head normalization via NI .
- 3 A combinatorial proof for that characterization.
- 4 How to extract quantitative information from derivations in NI .

Questions?



- ❶ Find all the derivations of $x : M \vdash xx : C$, for any linear type C and any multi type M .
- ❷ Find all the derivations of $x : M, y : N \vdash xy : C$, for any linear C and any multi M, N .
- ❸ Prove that all derivations in NI for $(\lambda x.xx)\lambda y.y$ have the form $\mathcal{D}_A^{\delta, l}$ shown on p.8, for any linear type A .
- ❹ Prove that there is no derivation of $\vdash (\lambda x.xx)\lambda y.yy : C$, for any linear type C .
- ❺ Find a derivation of $\vdash \lambda a.\lambda f.f(aaf) : C$, for some linear type C .
- ❻ Find a derivation of $\vdash (\lambda a.\lambda f.f(aaf))\lambda a.\lambda f.f(aaf) : C$, for some linear type C .
- ❼ Find all the derivations of $\vdash \lambda a.\lambda f.f(aaf) : C$, for any linear type C .
- ❽ Find all the derivations of $\vdash (\lambda a.\lambda f.f(aaf))\lambda a.\lambda f.f(aaf) : C$, for any linear type C .
- ❾ Prove rigorously the two lemmas on p. 13 and the two lemmas on p. 16.
- ❿ Prove rigorously the quantitative subject reduction (p. 15) and expansion (p. 17), by induction on the definition of $t \rightarrow_{h\beta} t'$ (see Day 3, p. 9).

Bibliography

- For an (almost gentle) introduction to non-idempotent intersection types:



Antonio Bucciarelli, Delia Kesner, Daniel Ventura. *Non-Idempotent Intersection types for the Lambda-Calculus*. Logic Journal of the IGPL, vol. 25, issue 4, pp. 431–464, 2017. <https://doi.org/10.1093/jigpal/jzx018>

- For a very advanced study about non-idempotent intersection types:



Beniamino Accattoli, Stéphan Graham-Lengrand, Delia Kesner. *Tight typings and split bounds, fully developed*. Journal of Functional Programming, vol. 30, 14 pages, 2020. <https://doi.org/10.1017/S095679682000012X>